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Continuity of Percolation Probability on Hyperbolic Graphs

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Let T_k be a forwarding tree of degree k where each vertex other than the origin has k children and one parent and the origin has k children but no parent $(k \geqslant 2)$. Define G to be the graph obtained by adding to T_k nearest neighbor bonds connecting the vertices which are in the same generation. G is regarded as a discretization of the hyperbolic plane H^2 in the same sense that Z^d is a discretization of R^d. Independent percolation on G has been proved to have multiple phase transitions. We prove that the percolation probability $\theta(p)$ is continuous on $[0,1]$ as a function of p.

KEY WORDS: Percolation; percolation probability; hyperbolic graphs.

1. INTRODUCTION

Let T_k be a forwarding tree of degree k, where each vertex other than the origin has k children and one parent and the origin has k children but no parent $(k \geqslant 2)$. Define G to be the graph obtained by adding to T_k nearest neighbor bonds connecting the vertices which are in the same generation (see Fig. 1). Independent percolation on the hyperbolic graph G was first studied by Benjamini and Schramm.[2] The name *hyperbolic graph* comes from the fact that G can be regarded as a discretization of the hyperbolic plane H^2. It was proved in ref. 2 that for independent percolation on G there exists no, infinitely many, or a unique infinite clusters, respectively when the parameter p is small, intermediate, or close to 1 (see also ref. 7 for results of Ising/Potts models on G). In order to make our statement precise, we first introduce a few notations. Independently declare each site of G to be open with probability p and closed with probability $1-p$.

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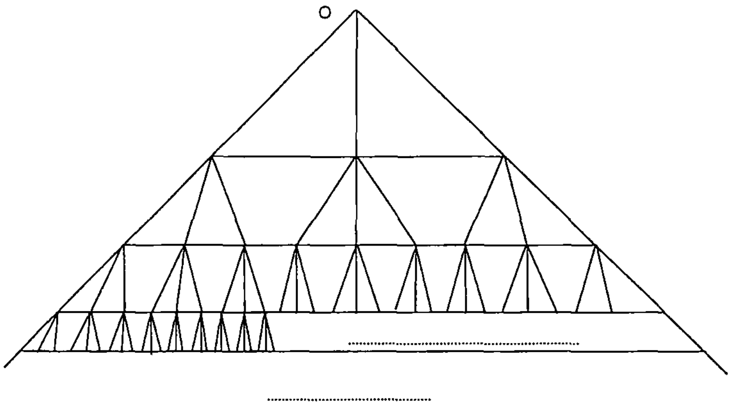

Fig. 1. G is obtained by adding T_3 horizontal nearest neighbor bonds connecting equal-generation sites of T_3.

Write P_p for the resulting probability measure and E_p the expectation. For any set of sites $A \subset G$ and any site $x \notin A$ denote by $x \leftrightarrow A$ the event that there exists a sequence of distinct sites $y_0, y_1,..., y_n$ such that $y_0 = x$ and $y_n \in A$, and for any $1 \leqslant i \leqslant n$, y_{i-1} and y_i are nearest neighbors and y_i is open. Note that for convenience y_0 is not required to be open (but y_n is). Denote by $o \leftrightarrow \infty$ the event that the above-defined sequence is infinite with $y_0 = o$, the origin. Define

$$\theta(p) = P_p(o \leftrightarrow \infty)$$

and

$$p_c = \inf\{p > 0 \colon \theta(p) > 0\}$$

Let ∂B_n be the set of sites in the nth generation of o (so $|\partial B_n| = k^n$) and let $B_n = \bigcup_{k=0}^{n} \partial B_k$, where $\partial B_0 = \{o\}$. For any site $x \in G$, denote by $x + B_n$ the shift of B_n by x, and by $x + \partial B_n$ the shift of ∂B_n by x, which represents the set of sites in the nth generation of x. When it will not cause confusion, we will also use B_n to denote the set of bonds which have both end points in B_n. Write $x + G$ for the shift of G by x, which represents all of the descendants of x. Now, $x + G$ is isomorphic to G.

For $p < p_c$, $\theta(p) = 0$ and hence it is a continuous function of p. For $p \geqslant p_c$, $\theta(p)$ is continuous from the right by a simple argument of Russo[6] (see also ref. 4, p. 118). Russo's argument is as follows. $\theta(p)$ is the limit of the decreasing sequence $P_p(o \leftrightarrow \partial B_n)$ as $n \to \infty$. Now, $P_p(o \leftrightarrow \partial B_n)$ is a continuous function of p since the event $o \leftrightarrow \partial B_n$ depends only on the status of the finitely many sites in B_n. So $\theta(p)$ is upper semicontinuous, hence it is continuous from the right since it is a nondecreasing function of p. For continuity from the left, it was proved by van den Berg and Keane[3] (see also ref. 4, p. 119) that if p is *strictly* above p_c and if the infinite cluster is *unique*, then $\theta(p)$ is continuous from the left. However, this method does not work if p is in the region where there are infinitely many infinite clusters or if $p = p_c$. In this note we use an argument similar to that of Barsky *et al.*[1] and that of Pemantle[5] to prove that for any $p \geqslant p_c$, $\theta(p)$ is continuous from the left. We therefore have the following theorem.

Theorem. For independent percolation on the hyperbolic graph G, $\theta(p)$ is a continuous function of p on $[0, 1]$. In particular, $\theta(p_c) = 0$.

For any site $x \in G$ write $\theta^x(p) = P_p(x \leftrightarrow \infty)$. For different sites x and y, $\theta^x(p)$ and $\theta^y(p)$ may be different functions since the graph G is inhomogeneous. But it is not hard to see by the FKG inequality that for any p either $\theta^x(p) = 0$ for all $x \in G$ or $\theta^x(p) > 0$ for all $x \in G$. It can be shown using the same argument presented in the next section that $\theta^x(p)$ is continuous in $[0,1]$ for any $x \in G$.

2. PROOF OF THEOREM

Define

$$Y_n = \{ y \in \partial B_n : o \leftrightarrow y \text{ in } B_n \}$$

Denote by $|Y_n|$ the number of sites in Y_n. We have the following result.

Lemma 1. $\lim_{n \to \infty} |Y_n| = \infty$ a.s. on the event $o \leftrightarrow \infty$.

The proof of the lemma is not difficult. If there exists a subsequence $\{Y_{n_k}\}$ such that $|Y_{n_k}|$ stays bounded, then the probability that none of the sites in Y_{n_k} is connected to ∞ is bounded away from zero, hence eventually $|Y_{n_k}| = 0$, a contradiction to $o \leftrightarrow \infty$. For a detailed argument see p. 122 of ref. 1.

Lemma 2. If $\theta(p) > 0$, then there exists $\delta > 0$ such that $\theta(p - \delta) > 0$.

Proof. For any number $A \in (0, \theta(p))$, choose M such that $M > 1/A$. From Lemma 1, $P_p(|Y_n| > M) \to \theta(p)$ as $n \to \infty$. So one can choose an integer $N = N(M, A, p)$ such that $P_p(|Y_N| > M) > A$. Now, $P_p(|Y_N| > M)$ is a continuous function of p since the event $|Y_N| > M$ depends only on the status of the finitely many sites in B_N. So one can choose $\delta > 0$ so that

$$P_{p-\delta}(|Y_N| > M) > A \tag{1}$$

Now fix the site density to be $p-\delta$. For each $x \in Y_N$, define

$$Y_N(x) = \{y \in x + \partial B_N : x \leftrightarrow y \text{ in } x + B_N\}$$

For different x and y of Y_N, $|Y_N(x)|$ and $|Y_N(y)|$ are i.i.d random variables having the same distribution as $|Y_N|$. So we have defined a Galton–Watson process which is supercritical since, by (1), $E|Y_N| \geqslant E|Y_N| I_{|Y_N| > M} \geqslant MA > 1$. So the probability that the above defined Galton–Watson process survives is positive. The proof is then completed by noticing that the percolation process with site density $p-\delta$ dominates the Galton–Watson process in the sense that if the Galton–Watson process survives, then $o \leftrightarrow \infty$.

An immediate consequence of Lemma 2 is that $\theta(p_c)=0$, since if $\theta(p_c)>0$, then $\theta(p_c-\delta)>0$ for some $\delta>0$, a contradiction to the definition of p_c.

Proof of the Theorem. As explained in the introduction, we only need to prove that $\theta(p)$ is continuous from the left. If $\theta(p)=0$, then $\theta(p)$ is clearly continuous from the left at p. Assume $\theta(p)>0$. By Lemma 2 there exists $\delta>0$ such that $\theta(p-\delta)>0$. For any $\varepsilon>0$ choose an integer M large enough such that $(1-\theta(p-\delta))^M<\varepsilon$. This inequality is still valid if δ is replaced by δ' with $0<\delta'\leqslant\delta$ since $\theta(p)$ is a nondecreasing function. As in the proof of Lemma 2, for the above chosen M, there exists a positive integer N such that $P_p(|Y_N|>M)>\theta(p)-\varepsilon$. By continuity of $P_p(|Y_N|>M)$ as a function of p, there exists $\delta_0>0$ such that $P_{p-\delta'}(|Y_n|>M)>\theta(p)-\varepsilon$ when $\delta'<\delta_0$. Hence we have that when $\delta'<\min(\delta_0,\delta)$,

$$\begin{aligned}
&\theta(p-\delta') \\
&\quad = P_{p-\delta'}(o\leftrightarrow\infty) \\
&\quad \geqslant P_{p-\delta'}(|Y_N|>M, \text{ and there exists } x\in Y_N \text{ such that } x\leftrightarrow\infty \text{ in } x+G) \\
&\quad \geqslant P_{p-\delta'}(|Y_N|>M)[1-(1-\theta(p-\delta'))]^M \qquad \text{by independence} \\
&\quad > (\theta(p)-\varepsilon)(1-\varepsilon) \geqslant \theta(p)-2\varepsilon
\end{aligned}$$

So $\theta(p)-\theta(p-\delta')<2\varepsilon$. This completes the proof, since ε is arbitrary.

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